Optimal Certainty in Kidney Exchange

Liam Connor

January 2025

Abstract

This paper considers a market with two exchange mechanism and agents who may or may not benefit from exchange. Agents arrive with an observable likelihood that they would benefit from exchange. Based on the observed value, a social planner decides to admit or reject the agent from entering the market. If admitted, the agent stochastically participates in one of two mechanisms: either they provide their own object to exchange, or they are allocated one from a limited supply. The planner aims to maximize the number of agents that benefit from exchange by establishing a minimum certainty threshold for participation. Agents do not know their type and cannot act strategically. The motivating example for this paper is kidney transplantation. Our results suggest that eligibility requirements for participating in both kidney exchange programs and the deceased donor list should be lowered. Adopting the papers suggestions would result in an additional 3,100 beneficial transplantations per year.

1 Introduction

1.1 Background

Each year, approximately 140,000 Americans are diagnosed with end stage renal disease (ESKD). ESKD is characterized by the near-complete loss of function of the kidneys. Patients with ESKD must either undergo dialysis or receive a kidney transplantation to survive. Compared to dialysis, transplantation is associated with a higher quality of life and reduced mortality. However, not all patients are suitable to receive kidney. For many patients, the risks associated with the kidney transplantation surgery outweigh the expected benefit. Of the 140,000 new ESKD diagnosis per year, less than one third of patients are ever enrolled on the kidney transplant waitlist. For many patients, the decision on whether to enroll them is clear. For others, there is uncertainty on whether they should be waitlisted. This paper considers the perspective of a public health official who is setting a 'certainty threshold' policy with the goal to maximize the number of beneficial transplantations. The certainty threshold is the minimum certainty that a patient would benefit from a transplant necessary to enter the waitlist. If the threshold is too high then matches will never be made, and donated kidneys will fail to find a match. If the threshold is set too low, then there is a risk that patients who are not expected to benefit will receive kidneys instead of those who are expected to benefit. To find the optimal threshold, we first provide background on the state of kidney disease and transplantation in the United States. Next, we introduce a model of the current US transplantation market, split between living donors and deceased donors. Using the introduced model, we estimate the optimal certainty threshold that should be adopted to maximize the number of successful transplantations. Finally, we compare these results to the status quo to calculate the overall impact on welfare.

Patients who receive transplantations have significantly better health outcomes than patients who remain on dialysis. Dialysis patients faced an all-cause mortality rate of 0.188 per person year, compared to 0.074 for transplant patients (USRDS 2023).[1] Healthcare costs were also significantly lower, at an average of $43,913 per person per year for transplant recipients, as opposed to $99,325 in 2021 for Medicare beneficiaries (USRDS 2023). However, the supply of donor kidneys limits the impact of transplantation. As of January 2025, there are over 90,000 patients enrolled on the kidney transplantation waitlist. On average, 17,177 patients received transplants per year[2], while 40,823 were added to the waitlist. The median waiting time is estimated at 4.05 years as of 2021 (Stewart, Mupfudze, and Klassen 2023), and an estimated 17 people die each day while awaiting a kidney transplant (Health Resources and Services Administration 2024).

Despite the positive outcomes associated with transplantation, less than a third of

[1]Due to selection bias, direct comparisons are of limited usefulness. Healthier and younger patients are more likely to be selected to waitlisted. However, trends hold when comparing patients who are waitlisted and receive a transplant, against those who are waitlisted and do not receive a transplant

[2]These statistics are found in the (OPTN 2024) database. Unless specified, data corresponds to the average from 2017 through 2021

patients are ever placed on the waitlist.[3] There are a variety of factors that contribute to this waitlist enrollment rate. First, some eligible patients choose not to enroll. Second, profit incentives may change the likelihood of nephrologists and/or dialysis providers recommend transplantation (Gander et al. 2020). Third and most prevalent, medical ineligibility makes many patients unlikely to benefit from transplantation. Kidney transplantation is stressful to the body and often requires the extended use of immunosuppressants. For elderly patients, or patients with serious comorbidities, the acute risks associated with transplantation may outweigh the long-term benefits. Additionally, a patient may be ineligible because of compliance or psychological concerns. The criteria used to evaluate eligibility are not firmly delineated and significant variation exists between practices (Batabyal et al. 2012). An elderly patient with an aggressive cancer may have a clearly negative expected outcome and an otherwise healthy young adult may have a clearly positive expected outcome. Between the extremes, however, there exists a grey area of uncertainty when evaluating the expected outcome of a transplant. This paper seeks to provide guidance on where to draw the line to maximize transplants with a positive outcome.

The majority of donor kidneys are sourced from deceased donors (USRDS 2023). To receive a donor kidney, patients must first register on the deceased donor list (DDL). Based on attributes such as age and blood type, patients are assigned a point value representing their priority (OPTN 2025). When an organ is harvested for donation, the highest priority patient on the list is contacted. A patient can either accept or reject an offered organ.[4] If rejected, the organ is offered to the patient with the next highest priority on the DDL. Harvested organs degrade with time and are eventually discarded if there is no match (Reese et al. 2021). Despite the high demand, 24% of harvested kidneys went unused in 2021 USRDS 2023. In total, an average of 17,171 kidneys were transplanted from deceased donors into patients undergoing dialysis per year between 2017 and 2021 OPTN 2024.

Living donors accounted for 26% of all donations between 2017 and 2021 OPTN 2024. Most people have two kidneys but only require one. In living donor donation, one kidney

[3]This is calculated as the number of new ESKD transplant candidates per year, divided by the number of new ESKD patients. (OPTN 2024)

[4]Donor organs are evaluated on compatibility, function, characteristics of the donor, and the expectation of future offers, among other considerations.

from the donor is transplanted into the recipient. The donor is typically a family member or friend of the recipient. However, in the case that the donor is incompatible with their preferred donor, the donor/patient pair can enter a kidney exchange market (KE). There are multiple kidney exchange programs within the United States that range in size, hospital participation, and matching methodology (Ashlagi and A. Roth 2021). In most kidney exchange markets, participants must be registered to the DDL before participating in kidney exchange, and are thus subject to the same restrictions and screening (OPTN 2025).

To motivate the paper, we consider two real-world scenarios. First, consider the case of a dialysis provider that signs a value based care contract with a health insurance company. In the contact, the dialysis provider is paid a bonus for enrolling more patients on the kidney transplantation waitlist.[5] Will this financial incentive improve welfare by enrolling patients who would benefit from transplantation but would otherwise not be enrolled? Or will it hurt welfare by overpopulating the waitlist with patients who are unlikely to benefit? The next scenario is of a husband with kidney failure and a wife who is a willing donor. In this scenario, the wife is not a match for her husband and a behavioral concern (such as alcoholism or medical non-compliance) prevents the husband from enrolling in the DDL and thus KE. How would the participation of the couple in the KE impact welfare?

We present two primary findings. First, in the KE market, no patient/donor pair should be restricted from entering (i.e. the threshold should be set to zero). In the DDL, the threshold should be lower than the status quo (although above zero). We find that lowering the status quo threshold to the optimal would lead to an additional 3,100 beneficial transplantations per year.

1.2 Literature Review

Kidney exchange is well-studied within the economic literature. The majority of recent papers modify a standard model of dynamic exchange. In the standard model, patient/donor pairs enter the market at a Poisson rate. The market is modeled as an Erdős–Rényi random graph with nodes that representing a single patient/donor pair, and

[5] While I cannot find a public source for this, I can personally verify this is true for at least one major dialysis provider.

directed edges representing compatibility between the the donor of one pair and the patient of another. Matching algorithms search this graph for cycles of two, cycles of three, or chains of multiple patients headed by an altruistic donor. Once a match is found, all donor/patient pairs are removed from the market. Welfare is defined as the average waiting period. Building from this model, researchers have considered the efficacy of different matching algorithms ((A. E. Roth, Sonmez, and Unver 2004), (Ünver 2010), (Ashlagi, Nikzad, and Strack 2023), among others); the importance of market thickness (Ashlagi, Nikzad, and Strack 2023); loss models where nodes leave the graph without matching at a Poisson rate (Akbarpour, Li, and Gharan 2020); the makeup of the market between of easy- and hard-to-match pairs (Ashlagi, Burq, et al. 2019); and operational aspects of kidney exchange, such as declined transplants (Dickerson, Procaccia, and Sandholm 2019); among others.

This paper builds directly from the KE modeling results found in (Akbarpour, Combe, et al. 2024), (Akbarpour, Li, and Gharan 2020), and (Ashlagi, Nikzad, and Strack 2023). Our paper is focused on interaction between the KE and DDL markets, not the structure of the KE market itself. Because of this, the model is designed to be compatible with all algorithms and extensions common in the KE literature. To use results from the literature, we first need to introduce a few common assumptions. The first assumption is that every donor/patient pair is equally likely to be compatible with another pair.[6] Second, the matching algorithm is myopic. A myopic algorithm is a common assumption in literature, such as in (Akbarpour, Combe, et al. 2024) and (Ashlagi, Nikzad, and Strack 2023). Third, we assume that past a given threshold, increasing the rate of entry in KE has near-zero impact on expected wait times. This is shown in (Akbarpour, Combe, et al. 2024). Fourth, we implicitly model loss. To support the continued relevancy of literature results under this formulation, we reference (Akbarpour, Li, and Gharan 2020). Fifth, we expect the KE market to reach a steady state. That is, as time runs to infinity, the pool size and exit rates will stabilize around an expected value. Ultimately, the results we yield are robust under all state-of-the-are matching algorithms and any market makeup.

This paper departs from literature by integrating the DDL and endogenously defining the market entry rate through the certainty threshold. Most papers focus exclusively

[6] In-fact, we only need that a patient's expectation of benefiting from exchange is independent of their compatibility with others.

on the kidney exchange market. In (Ashlagi and A. Roth 2021), authors Itai Ashlagi and Alvin Roth call attention to integrating the DDL into the KE as a promising area of future study. One example of this integration effort is found in (Akbarpour, Combe, et al. 2024), in which the authors consider drawing from the DDL to start chains in the absence of altruistic donors. Given current legal and regulatory hurdles, this integration is not immediately actionable. Our model seeks to reflect the current US market when considering the interaction between the DDL and KE. Additionally, all models that we are aware of consider entry rate exogenous. In contrast, entry rate is defined endogenously as a function of the chosen threshold.

2 Model

2.1 Setup

Agents enter the a population pool with a Poisson arrival rate given by n. Each agent has an unobservable true type, $x_i \in \{0, 1\}$, which corresponds to their payoff from matching. Each agent also has an observable likelihood that they would benefit from matching, given by $\xi_i = \Pr[x_i = 1] \in [0, 1]$. In our context, 'benefit from matching' is defined as the outcome of transplantation is preferable to continued treatment on dialysis. Agents are otherwise identical. Let ξ_i be drawn from a distribution given by $g(\xi)$, where $g(\xi)$ is a positive-valued, continuous, and well-behaved probability density function.

Let $\delta \in [0, 1]$ represent the threshold established by the social planner. If $\xi_i \geq \delta$, then the agent is permitted to enter a market. After being accepted by the social planner, the agent then joins one of two markets. With probability $\gamma \in [0, 1]$, the agent enters the kidney exchange market, denoted KE. With probability $(1 - \gamma)$, the agent instead enters the deceased donor market, denoted DDL. Note that KE refers to all live-donor transplantations, not just donations that occur through the kidney exchange.[7] Also note that patients can enter either the DDL or the KE, but not both.[8]

Moving forward, we will be evaluating the model once it reaches steady state. The

[7]This simplification is justified by the fact that we are assuming KE entry rate is already high enough to not impact average wait time, and patients who receive a live donation do not use a kidney from the DDL.

[8]In the real world, enrolling in the DDL is often a prerequisite for joining a KE program. However, we assume patients with access to a live donor do not use the DDL.

model will stabilize around certain entry rate, market size, and exit rate.[9] Values reflect the long-run expected value, although specific instances will vary.

By construction, each market has a Poisson arrival rate, λ, given by

$$\lambda_{KE} = n\gamma \int_{\delta}^{1} g(\xi)\, d\xi$$

$$\lambda_{DDL} = n(1-\gamma) \int_{\delta}^{1} g(\xi)\, d\xi$$

for KE and DDL respectively. Additionally, each market has a matching rate, given by the positive valued, monotonically increasing functions $h_{KE}(\lambda_{KE})$ and $h_{DDL}(\lambda_{DDL})$. Define $h = h_{KE} + h_{DDL}$ The function h captures the expected number of agents matched per period. The forms of h_{KE} and h_{DDL} are presented in Assumptions 2 and 3 respectively.

We define a new function that gives the precision of the social planner, $p(\delta)$. In other words, $p(\delta)$ captures the expected value of matching a permitted agent. This function is given by,

$$p(\delta) = \frac{\int_{\delta}^{1} g(\xi)\xi d\xi}{\int_{\delta}^{1} g(\xi) d\xi},$$

which is simply the expected value of $g(\xi)$, truncated to match the threshold.

Finally, we introduce a measure of welfare, $w(\delta)$. We define welfare as the rate that patients that benefit from matching ($x_i = 1$) that are matched in steady state. This is given by

$$w(\delta) = p(\delta)(h_{KE}(\lambda_{KE}) + h_{DDL}(\lambda_{DDL})) \tag{Total welfare}$$

In most kidney exchange papers, such as (Ashlagi, Burq, et al. 2019), (Akbarpour, Combe, et al. 2024), and (Ünver 2010)[10], welfare measures the average waiting time of agents after entering the KE. In these models, patients remain in the waiting pool until matched. Once matching has reached steady state, departure rate is equivalent to arrival rate. Little's Law[11] tells us that waiting time and arrival rate are directly linked, and therefore minimizing average wait time is equivalent to maximizing departure rate. One notable exception is found in (Akbarpour, Li, and Gharan 2020). In the model they present, agents are allowed to depart without being matched, known as loss. This corresponds to the patient dying or becoming too sick to receive a transplant. The authors

[9] A full treatment of steady state, and proof that it will be reached in a loss model is given in (Ashlagi, Burq, et al. 2019)

[10] Ünver minimizes time-discounted wait time

[11] The average number of items in a system is equal to the arrival rate multiplied by wait time

tweak the welfare function to minimize loss rate instead of waiting time. In our model, loss rate is implicitly defined as $(n - w(\delta))$. This is the number of patients who do not receive a beneficial match per period in steady state. Therefore, our objective function aligns with the KE literature and accurately models the situation.

2.2 General Analysis

We now turn to understanding the relation between the threshold and expected matches. We begin by finding the threshold maximizing function for a single market.[12] Let w_S denote single market welfare. To find the FOC, we must solve:

$$\begin{aligned} w_S(\delta) &= p(\delta)h(\lambda) \\ \frac{dw_S}{d\delta} &= \frac{dp}{d\delta} \cdot h(\lambda) + p(\delta) \cdot \frac{dh}{d\lambda} \cdot \frac{d\lambda}{d\delta} = 0 \\ \underbrace{\frac{dp}{d\delta} \cdot h(\lambda)}_{\textit{Benefit from increased precision}} &= \underbrace{-p(\delta) \cdot \frac{dh}{d\lambda} \cdot \frac{d\lambda}{d\delta}}_{\textit{Detriment from reduced market size}} \end{aligned} \tag{1}$$

It is beneficial to develop intuition for Eq. (1) before progressing. We know that $p'(\delta)$ is nonnegative, as a more selective social planner is more precise. Similarly, $h(\lambda)$ is nonnegative by construction. The LHS is therefore a nonnegative value that captures how welfare improves from increased precision. On the RHS, $h'(\lambda)$ is nonnegative by construction, as higher agent entrance rate corresponds to a higher rate of matching. In contrast, $\lambda'(\delta)$ is negative, as increasing the threshold reduces the number of agents that are enter the market. Therefore, the RHS captures the reduction in welfare from a reduced matching population. Putting it together, Eq. (1) is telling us that optimality is achieved when the gains from increased precision is equivalent to the losses from reduced market size.

Using Leibniz's rule, we can calculate $\frac{dp}{d\delta}$:

$$\begin{aligned} p(\delta) &= \frac{\int_\delta^1 g(\xi)\xi d\xi}{\int_\delta^1 g(\xi)d\xi} \\ \frac{dp}{d\delta} &= \frac{g(\delta)(\int_\delta^1 g(\xi)\xi d\xi - \delta \int_\delta^1 g(\xi)d\xi)}{(\int_\delta^1 g(\xi)d\xi)^2} \end{aligned} \tag{2}$$

[12]In this subsection, we drop the subscript for λ and h as the analysis is the same between markets. Note we use γ, although the analysis is the same for the DDL if γ is replaced by $(1 - \gamma)$.

Next we calculate $\frac{d\lambda}{d\delta}$:

$$\lambda(\delta) = n\gamma \int_{\delta}^{1} g(\xi)\, d\xi$$
$$\frac{d\lambda}{d\delta} = -n\gamma g(\delta) \tag{3}$$

Plugging (2) and (3) into (1), we the following FOC:

$$\frac{dw_S}{d\delta} = \frac{g(\delta)(\int_\delta^1 g(\xi)\xi d\xi \cdot (h(\lambda) - h'(\lambda)n\gamma \int_\delta^1 g(\xi)d\xi) - h(\lambda)\delta \int_\delta^1 g(\xi)d\xi}{(\int_\delta^1 g(\xi)d\xi)^2} = 0 \tag{4}$$

Eq. 4 gives the optimality condition for a single market. The equation clearly depends on assumptions we make about the function h and distribution g. We introduce those assumptions in the following section.

2.3 Specific Threshold Analysis

To specify the model to kidney exchange, we introduce the following three assumptions:

Assumption 1: $g(\xi)$ *is uniformly distributed.* This simplifies the math, and is relaxed in the Extensions section.

Assumption 2: $h_{DDL}(\lambda_{DDL}) = c(1 - e^{-k\lambda_{DDL}})$ *for some* $c \in \mathbb{R}^+$, $k \in \mathbb{R}$. The variable c corresponds to the number of deceased donor kidneys harvested and k captures the efficiency of utilizing harvested kidneys. This functional form was chosen to match a set of empirical observations about the deceased donor list. First, $h_{DDL}(0) = 0$. Second, $\lim_{\lambda\to\infty} h = c$, as the maximum number of kidneys distributed is capped by the number harvested. Third, h is monotonically increasing with the waitlist entry rate. A higher entry rate corresponds to a longer waitlist, a longer waitlist increases the number of patients a harvested kidney can be offered to before it degrades, and an increased number of offerings improves the likelihood that any patient accepts the harvested kidney.

Assumption 3: $h_{KE}(\lambda_{KE}) = \alpha\lambda_{KE}$ *for some* $\alpha \in (0, 1)$. This assumption follows from both theoretical and empirical analysis. The observed probability of benefiting from exchange does not change the likelihood that an agent's good is compatible with any agent. Therefore, each agent has an equal likelihood of contributing to a 2-cycle, 3-cycle, or chain, regardless of ξ_i. The entrance of an new agent corresponds to a expected number of matches, $\alpha > 0$, in steady state.[13] The value $(1-\alpha)$ corresponds to the portion of patients that die or are removed from the kidney exchange program while waiting on a match. Realistically, α is increasing with entry rate. We are assuming the entry rate is high enough to make this effect negligible.[14]

Using Assumption 3, we can derive the first major result of the paper. Note that Proposition 1 does not depend on Assumptions 1 and 2.

Proposition 1 *For any well-behaved, positive valued probability distribution, $g(\xi)$, and myopic matching algorithm, $w_{KE}(\delta)$ reaches a maximum at $\delta = 0$ and w_{KE} is weakly monotonically decreasing in δ once the system reaches steady state.*

Proof 1 *Under a myopic matching algorithm, new matches are considered immediately after the entrance of each new agent. Therefore, we can define the following equation to represent the successful matches made per period:*

$$w_{KE}(\delta) = \underbrace{\alpha}_{\textit{Matches per entrant}} \cdot \underbrace{n\gamma \int_{\delta}^{1} g(\xi)\, d\xi}_{\textit{Entrant frequency, } \lambda} \cdot \underbrace{\frac{\int_{\delta}^{1} g(\xi)\xi d\xi}{\int_{\delta}^{1} g(\xi) d\xi}}_{\textit{Portion of matches benefit agents, } p(\delta)}$$

Then we can derive FOC as follows:

$$\begin{aligned}
\max_{\delta} w(\delta)_{KE} &= \alpha n\gamma \cdot \cancel{\int_{\delta}^{1} g(\xi)\, d\xi} \cdot \frac{\int_{\delta}^{1} g(\xi)\xi d\xi}{\cancel{\int_{\delta}^{1} g(\xi) d\xi}} \\
&= \alpha n\gamma \cdot \int_{\delta}^{1} g(\xi)\xi d\xi \\
\frac{dw_{KE}}{d\delta} &= -\alpha n\gamma\delta^* g(\delta) = 0
\end{aligned}$$

Because $g(\delta)$ and all parameters are positive valued, this yields

$$\delta^* = 0$$

Also, note that $\frac{dw_{KE}}{d\delta} < 0$, so welfare is decreasing in δ.

[13] In lossless KE models, $\alpha = 1$. In loss models (see (Akbarpour, Li, and Gharan 2020)), expected matches per entrant stabilizes to some $\alpha \in (0, 1)$. Also note that this claim is robust under models that consider hard-to-match agents, as an agent's ξ_i is independent of their difficulty of matching.

[14] See Section 1.2 for more details.

The intuition behind this result is simple. When a new agent joins the market, they always bring an additional good. When a match is made, their good is transferred to another agent resulting in an expected payoff of their own (ξ_i) and the other agent (ξ_j). Therefore, even if they have a low likelihood of benefiting, they facilitate a match that would otherwise not happen.

The implications of Proposition 1 sharply contradict current kidney exchange policies. The full implications are explored in Section 3.

We now use Assumptions 1-3 to arrive at the optimal threshold level. First, we find precision and its derivative under the uniformity assumption:

$$p(\delta) = \frac{\int_\delta^1 g(\xi)\xi d\xi}{\int_\delta^1 g(\xi)d\xi} = \frac{1+\delta}{2} \tag{5}$$

$$\frac{dp}{d\delta} = \frac{1}{2} \tag{6}$$

Next, we calculate the optimal DDL threshold. Find the input rate as follows:

$$\lambda_{DDL}(\delta) = n(1-\gamma)\int_\delta^1 g(\xi)d\xi = n(1-\gamma)(1-\delta) \tag{7}$$

$$\frac{d\lambda_{DDL}}{d\delta} = -n(1-\gamma) \tag{8}$$

And the processing function:

$$\begin{aligned} h_{DDL} &= c(1-e^{-k\lambda_{DDL}}) \\ &= c(1-e^{-kn(1-\gamma)(1-\delta)}) \end{aligned} \tag{9}$$

$$\frac{dh_{DDL}}{d\delta} = ckn(1-\gamma)e^{-kn(1-\gamma)(1-\delta)} \tag{10}$$

Finally, we combine Eq. (5), (6), (9) and (10) under the FOC given by Eq. (1) to find δ^*_{DDL}, the optimal threshold for the DDL market:

$$\frac{dw_{DDL}}{d\delta} = \frac{1}{2}c\left(1-e^{-kn(1-\gamma)(1-\delta^*_{DDL})}\right) - \frac{1}{2}(1-\delta^*_{DDL})ckn(1-\gamma)e^{-kn(1-\gamma)(1-\delta^*_{DDL})} = 0 \tag{11}$$

$$e^{-kn(1-\gamma)(1-\delta^*_{DDL})} = \frac{1}{(1-\delta^*_{DDL})kn(1-\gamma)+1}$$

$$\delta^*_{DDL} = 1 - \frac{\ln\left(1+(1+\delta^*_{DDL})kn(1-\gamma)\right)}{kn(1-\gamma)} \tag{12}$$

Eq. (12) can be solve computationally.

From Proposition 1, we know that the optimal KE threshold, δ^*_{KE}, equals 0. For the sake of completeness, we show that Assumptions 1-3 align with this result:

$$\begin{aligned} h_{KE}(\lambda) &= \alpha\lambda \\ &= \alpha n\gamma \int_\delta^1 g(\xi)d\xi \\ &= \alpha n\gamma(1-\delta) \qquad (13) \\ \frac{dh_{KE}}{d} &= -\alpha n\gamma \qquad (14) \end{aligned}$$

Calculate δ^*_{KE} by plugging Eq. (5), (6), (13), and (14) into Eq. (1):

$$\begin{aligned} \frac{dw_{KE}}{d\delta} &= \frac{1}{2}\alpha n\gamma(1-\delta) - \frac{1}{2}\alpha n\gamma(1+\delta) = 0 \qquad (15) \\ 1-\delta &= 1+\delta \\ \delta^*_{KE} &= 0 \qquad (16) \end{aligned}$$

Per the Total Welfare equation, we simply add w_{KE} and w_{DDL} to find total welfare. This result is presented in the following proposition:

Proposition 2 *Under assumptions (1-3), the threshold that maximizes welfare is given by:*

$$\delta^* = 1 + \frac{1}{kn(1-\gamma)} \ln\left(\frac{1 - \frac{2\alpha n\gamma\delta^*}{c}}{1 + (1+\delta^*)kn(1-\gamma)}\right)$$

Proof 2 *Total welfare is given by*

$$\begin{aligned} w(\delta) &= p(\delta)(h_{KE}(\lambda_{KE}) + h_{DDL}(\lambda_{DDL})) \\ &= w_{KE} + w_{DDL} \\ \frac{dw}{d\delta} &= \frac{dw_{KE}}{d\delta} + \frac{dw_{DDL}}{d\delta} \qquad (17) \end{aligned}$$

We simply substitute (11) and (15) into (17) to find the FOC:

$$\frac{dw}{d\delta} = \frac{1}{2}c\left(\left(1 - e^{-kn(1-\gamma)(1-\delta)}\right) - (1-\delta)kn(1-\gamma)e^{-kn(1-\gamma)(1-\delta)}\right) - \alpha n\gamma\delta = 0$$

This can be rearranged to yield Proposition 2.

3 Welfare Effects

Adjusting the required certainty that a transplant will benefit the patient may have a significant impact on the number of beneficial kidney transplantations performed. We estimate the effects here. First, we empirically derive estimates for the model parameters. Variables n and c are directly reported. The cutoff, δ, is derived by the portion of ESKD patients who are waitlisted, assuming the uniformity of $g(\xi)$. Variables k, γ, and α are not directly reported, or are model simplifications. Estimating the variables involved two steps. First, an initial empirical estimate was made. Second, the model was fit to historical rates of transplantation. Estimates are listed in Table 3.[15]

Parameter	**Value**	**Description**
n	362.2	ESKD diagnosis, daily
c	59.4	Deceased donor kidneys harvested, daily
k	0.016	Deceased kidney utilization parameter
γ	0.156	Portion of patients entering the live donor market
α	0.955	Survival rate for patients enrolled in KE
δ	0.691	Waitlist eligibility cutoff

Table 1: Empirically derived model parameter estimates

Plugging these parameters into the model yields an estimated daily transplantation rate $h = 63.5$ (actual: 63.7), composed of a DDL rate $h_{DDL} = 46.9$ (actual: 47.0) and a KE rate $h_{KE} = 16.6$ (actual: 16.6). The cutoff $\delta = 0.691$ corresponds to an expected precision of 84%. This aligns with the actual 3-year survival rate of 84% (Ghelichi-Ghojogh et al. 2022) but ignores the complexities of our 'beneficial transplant' definition.

Lowering the certainty threshold to its optimal value would lead to an additional 3,100 beneficial transplantations per year. The current threshold value is set to $\delta = 0.69$, while the numerically solved optimal value is $\delta^* = 0.37$. This improves the total beneficial transplants from 54.0 to 63.2. This improvement is primarily driven by an elevated rate of KE transplants. The difference between the optimal and actual KE threshold is significant, as expected by Proposition 1. The optimal thresholds for the KE and DDL markets are given by $\delta^*_{KE} = 0$ and $\delta^*_{DDL} = 0.57$ respectively. See Figure 1 for a plot of $w(\delta)$. Also see Appendix B for a full table of results, detailing matching rates for each market at optimal and status quo thresholds.

[15]See Appendix C for sources and further discussion.

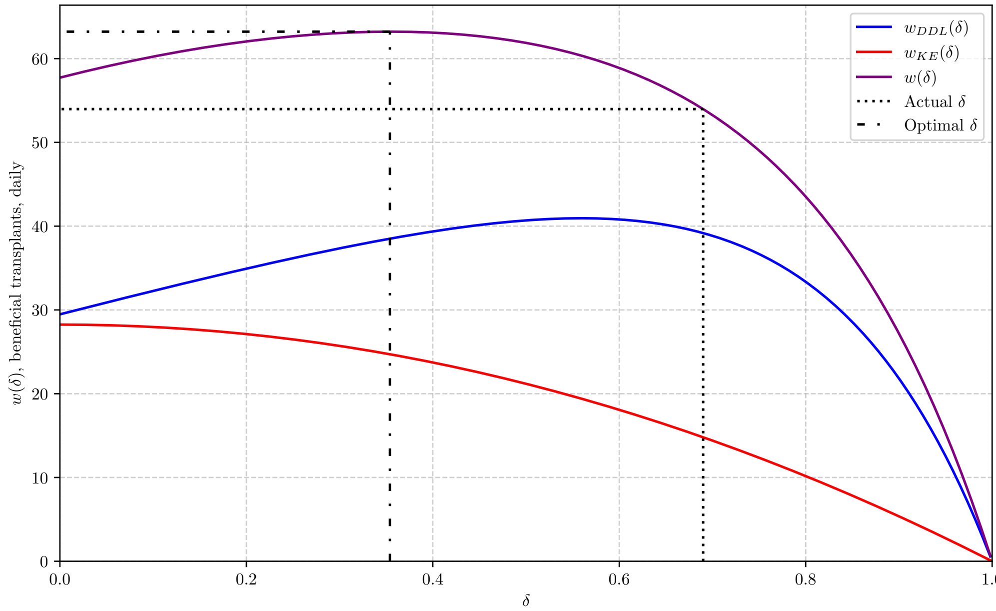

Figure 1: Welfare, $w(\delta)$ for uniform distribution.

4 Extensions

4.1 Vary the distribution of $g(\xi)$

The assumption that $g(\xi)$ is uniform is unrealistic. The true distributions is likely closer to bimodal, with most patients either clearly eligible or clearly ineligible. As a proof of concept, define a new distribution, $g_{bimodal}$, given by:

$$g_{bimodal}(\xi) = \begin{cases} \frac{0.4\,\phi_1(\xi)+0.6\,\phi_2(\xi)}{\int_0^1 (0.4\,\phi_1(\xi)+0.6\,\phi_2(\xi))\,d\xi}, & 0 \leq \xi \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

where the individual normal distributions are given by $\phi_1 \sim \mathcal{N}(0, 0.2)$ and $\phi_2 \sim \mathcal{N}(0.9, 0.2)$. The PDF and CDF of this function is presented Figure 2.

We solve non-uniform distributions of $g(\xi)$ computationally. We can find $\delta^* = 0.42$ computationally. As in the uniform distribution case, $\delta^* < \delta$, as $\delta = 0.84$ under this distribution. Given that Proposition 1 does not depend on the form of $g(\xi)$, we would expect this result to hold under a variety of $g(\xi)$ forms.[16] In every tested bimodal distribution, $\delta > \delta^*$ remained true. See Appendix B for additional graphs and values.

[16] As a reminder, we calculate δ such that $\int_0^\delta g(\xi)d\xi = 0.691$, corresponding to the fact that 30.9% of

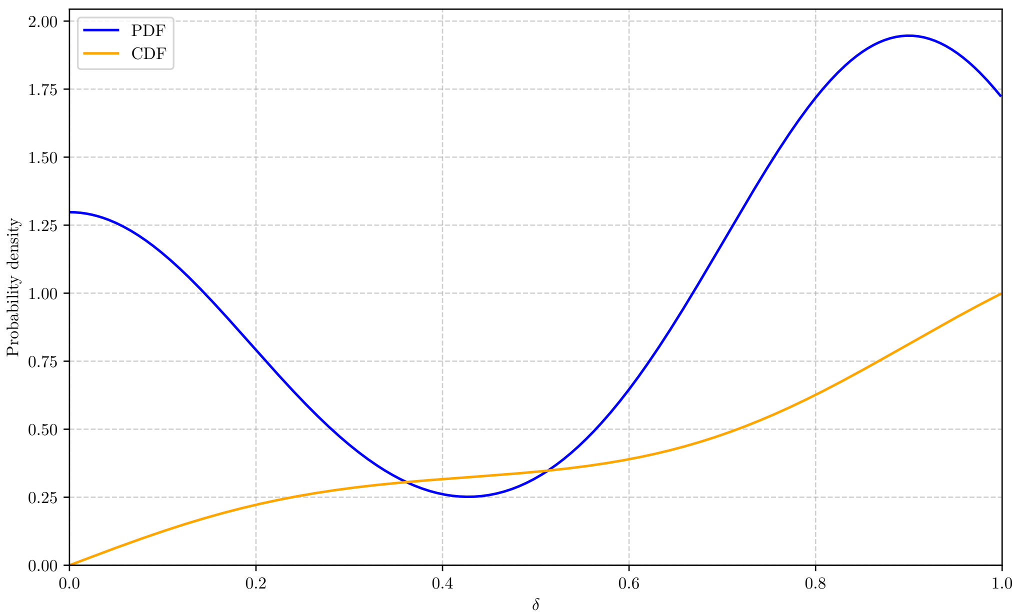

Figure 2: An example bimodal distribution of $g(\xi)$.

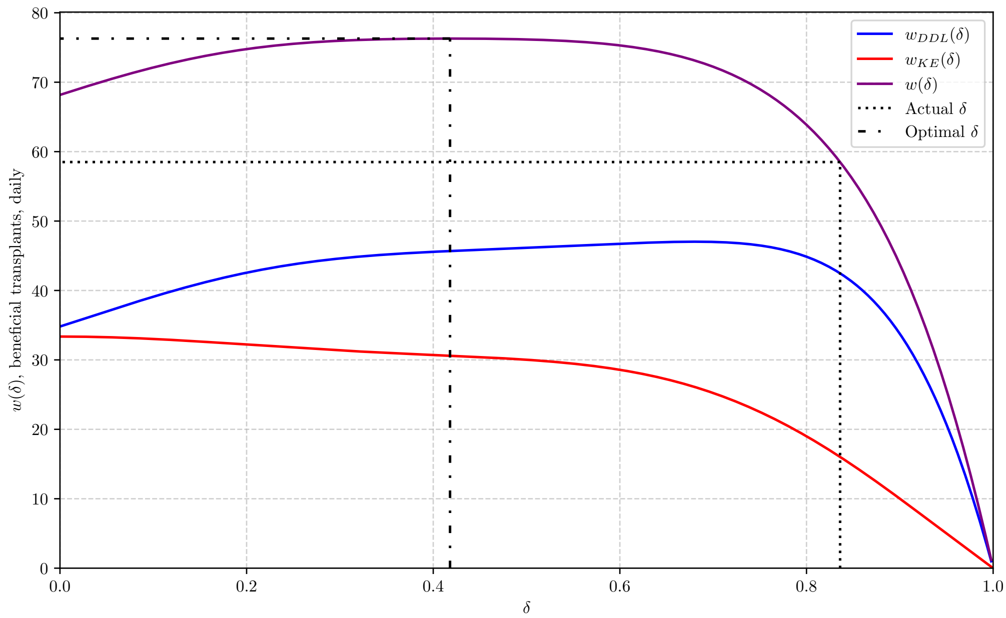

Figure 3: Welfare, $w(\delta)$ for bimodal distribution.

total ESKD patients are waitlisted.

4.2 Include the cost of non-beneficial transplantations

Currently, the welfare function only maximizes beneficial matches. This could easily be adapted to include a penalty for non-beneficial matches. While the opportunity cost of non-beneficial matches is already captured, this weight would reflect the cost of transplantation itself. To pursue this extension, let us define a new welfare function

$$\begin{aligned} w^{net}(\xi) &= p(\delta)(h_{KE}(\lambda_{KE}) + h_{DDL}(\lambda_{DDL})) - \mu(1-p(\delta))(h_{KE}(\lambda_{KE}) + h_{DDL}(\lambda_{DDL})) \\ &= ((1+\mu)p(\delta) - \mu)(h_{KE}(\lambda_{KE}) + h_{DDL}(\lambda_{DDL})) \end{aligned}$$

Where $\mu \in \mathbb{R}^+$ is a weight for non-beneficial matches. We can present an updated Proposition 1 accounting for μ:

Proposition 3 *For any well-behaved, positive valued probability distribution, $g(\xi)$, myopic matching algorithm, and $\mu \geq 0$ weight on non-beneficial matches, $w^{net}_{KE}(\delta)$ is maximized by*

$$\delta^*_{KE} = \frac{\mu}{1+\mu}$$

once the system reaches steady state.

Proof 3 *Building off the problem description presented in Proposition 1, we can define the KE model welfare function as:*

$$\begin{aligned} w^{net}_{KE}(\delta) &= \alpha \cdot \left((1+\mu)\frac{\int_\delta^1 g(\xi)\xi d\xi}{\int_\delta^1 g(\xi)d\xi} - \mu \right) \cdot n\gamma \int_\delta^1 g(\xi)d\xi \\ &= \alpha n\gamma \cdot \left((1+\mu) \cdot \int_\delta^1 g(\xi)\xi d\xi - \mu \int_\delta^1 g(\xi)d\xi \right) \end{aligned}$$

And the FOC is given by

$$\begin{aligned} \frac{dw^{net}_{KE}}{d\delta} &= \alpha n\gamma \cdot g(\delta^*) \cdot (\mu - \delta^*(1+\mu)) = 0 \\ \mu &= \delta^*(1+\mu) \\ \delta^* &= \frac{\mu}{1+\mu} \end{aligned}$$

As expected by Proposition 1, $\delta \to 0$ as $\mu \to 0$. Intuitively, the larger the weight on non-beneficial matches, the higher the optimal certainty threshold. It is also interesting to note that this result does not depend on distribution $g(\xi)$ or KE market efficiency α.

We apply computational methods to find optimal certainty thresholds and net welfare under both uniform and bimodal distributions. We assume $\mu = 1$ for the sake of simplicity, although this could easily be adjusted based on empirical costs. Refer to Appendix B for specific values.

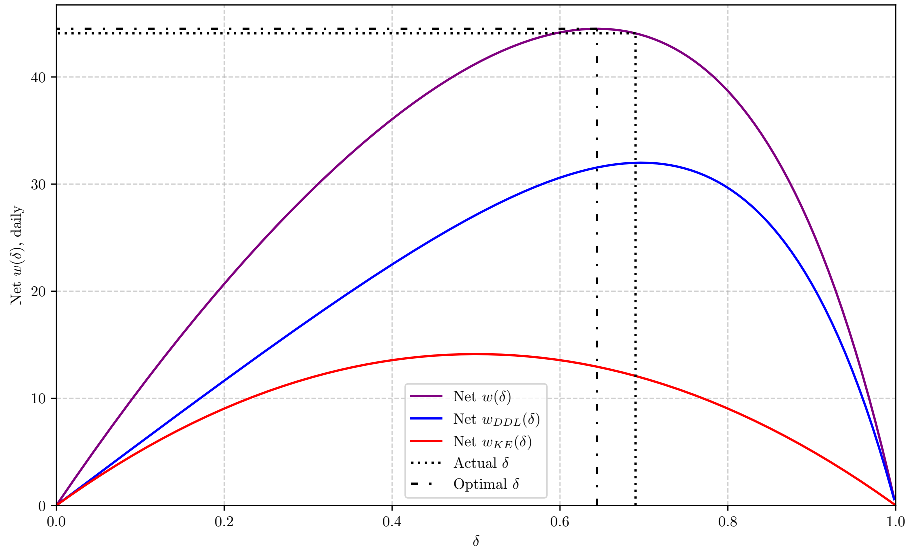

Figure 4: Net welfare, $w^{net}(\delta)$ for uniform distribution and $\mu = 1$.

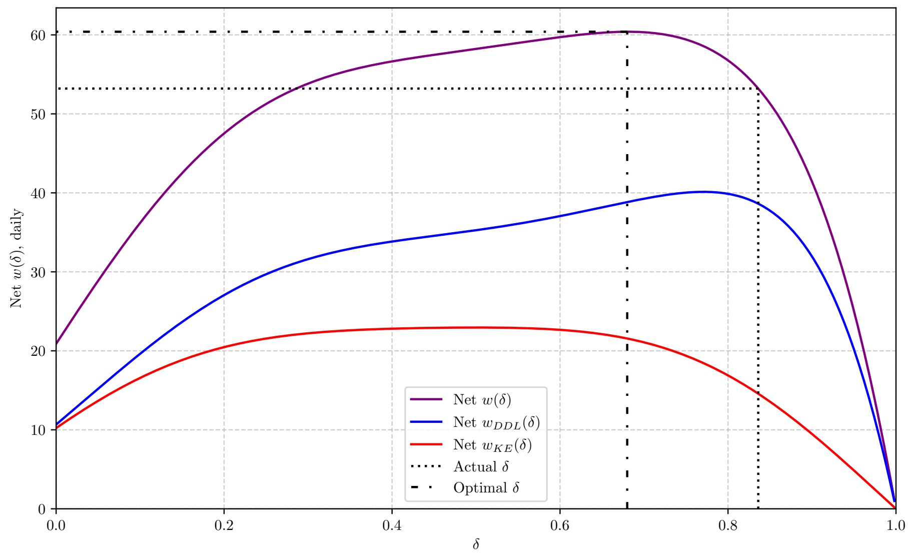

Figure 5: Net welfare, $w^{net}(\delta)$, for bimodal distribution and $\mu = 1$.

4.3 Allow δ to vary by market

Let the DDL and KE could have independent thresholds. We expect welfare to improve and $\delta^*_{DDL} > \delta^* > \delta^*_{KE} = 0$. In the base model (uniform distribution and $\mu = 0$), allowing the threshold to vary between markets results in an expected 68.2 beneficial transplantations, a 9.6% increase from the optimal single-threshold rate of 62.2.

4.4 Introduce patient agency

If a patient's doctor does not think a patient would benefit from receiving a transplant, i.e. $\xi_i < 0.5$, the patient will likely not join the waitlist even if they are permitted to. The likelihood a patient joins could be modeled as a function of their perceived likelihood of benefiting, ξ_i. This effect is likely stronger within the KE market, where a patient must privately find a donor to enter. While the optimal threshold for permission would remain at 0, we would expect many patients would choose not to join. This extension would depress the welfare effect expected by lowering the threshold, but would not eliminate it. Additionally, we expect the optimal threshold for the DDL to decrease.

5 Conclusion

In this paper, we consider a bifurcated market and agents that may or may not benefit from exchange. We adopt the position of a social planner who seeks to maximize the number of beneficial exchanges. When agents must provider their own good to participate, we find that there should be no minimum certainty requirement for joining the market. When goods are distributed from a limited supply, we find that the threshold should be set such that the marginal benefit from increased precision is equivalent to the marginal detriment from reduced market size.

When applied to the USA kidney transplantation market, we find that the optimal certainty threshold is significantly below the status quo certainty threshold. This finding remained robust on a variety of extensions. In direct contrast to current practice, we found that KE programs would benefit from removing restrictions on which patients can join.[17] We can now answer the two motivating real-world scenarios from the introduction. First, we expect that financial incentives aimed at increasing waitlist enrollment would

[17] Note that this paper relates to changing restrictions on patients, not donors.

improve welfare. Second, the example couple should be permitted to join a KE program, regardless of the husband's eligibility for the DDL.

The magnitude of impact and exact optimal level would require additional empirical study to calculate. It is clear, however, that a downward revision of the certainty threshold would improve the number of beneficial transplantations per year.

6 Acknowledgments

This work was supported in part by Health Resources and Services Administration contract HHSH250-2019-00001C. The content is the responsibility of the authors alone and does not necessarily reflect the views or policies of the Department of Health and Human Services, nor does mention of trade names, commercial products, or organizations imply endorsement by the U.S. Government.

This work acknowledges the use of artificial intelligence (AI) tools during the research and writing process. Specifically, AI was employed for tasks such as checking mathematics, graphing, LaTeX formatting, and writing this disclaimer. All interpretations, conclusions, and claims presented in this paper are the sole responsibility of the author and reflect their original contributions. Any errors or omissions are also the responsibility of the author.

I would like to thank Daria Soboleva for careful editing. I would also like to the thank Ryan Niedernhofer for reviewing a proof (despite the proof not making it into the final draft).

A Glossary

Term	Description
n	Poisson arrival rate to ESKD pool
x_i	Unobservable true type, captures whether agent i benefits from exchange
ξ_i	Observed likelihood an agent benefits from exchange, $\xi_i = \Pr[x_i = 1]$
$g(\xi)$	Population Probability distribution function of ξ_i
δ	Certainty threshold. $\xi_i \geq \delta$ is required to enter the market
KE	Kidney Exchange market. Abstracted to include all live donors
DDL	Deceased Donor List market
γ	Probability any agent enters the KE
$(1-\gamma)$	Probability any agent enters the DDL
λ_{KE}	Poisson entry rate into the KE market
λ_{DDL}	Poisson entry rate into the DDL market
h_{KE}	KE market processing function, the number of agents matched
h_{DDL}	DDL market processing function, the number of agents matched
$p(\delta)$	Precision, the expected value of matching a permitted agent
$w(\delta)$	Welfare, the rate of matched patients
c	Harvest deceased donor kidneys per period
k	Deceased donor kidney utilization parameter
α	Portion of patients who enter KE that are ultimately matched
μ	Weight value for non-beneficial matches

Table 2: Glossary of terms and symbols.

B Results and Figures

Distribution	**Uniform**		**Bimodal**	
Welfare function	**Base,** $\mu = 0$	**Net,** $\mu = 1$	**Base,** $\mu = 0$	**Net,** $\mu = 1$
δ^E	0.69	0.69	0.84	0.84
δ^*	0.35	0.64	0.42	0.68
δ^*_{DDL}	0.56	0.70	0.68	0.77
δ^*_{KE}	0.00	0.50	0.00	0.50
$w(\delta^E)$	53.98	44.07	60.33	53.21
$w_{DDL}(\delta^E)$	39.17	31.99	42.47	38.63
$w_{KE}(\delta^E)$	14.80	12.09	16.03	14.58
$w(\delta^*)$	63.22	44.50	76.29	60.40
$w_{DDL}(\delta^*)$	40.95	31.55	47.03	38.84
$w_{KE}(\delta^*)$	28.25	12.95	33.36	21.56
Welfare increase	9.24	0.43	17.79	7.19
Improvement (%)	17%	1%	30%	14%
Total transplants, h^E	63.88	63.88	63.79	63.79
Non-beneficial	9.90	9.90	5.29	5.29
(% beneficial)	85%	85%	92%	92%
Total transplants, h^*	93.38	69.10	95.61	95.88
Non-beneficial	30.16	12.30	19.32	12.74
(% beneficial)	68%	82%	80%	85%

Table 3: Optimal thresholds, transplants, and welfare with extensions.

Table 3 presents the optimal certainty thresholds, total transplants, and welfare improvements. Note that ^{E} signifies empirically estimated status quo. The following figures graph $w(\delta)$, $w^{net}(\delta)$, and $h(\delta)$ against uniform and bimodal distributions of $g(\xi)$.

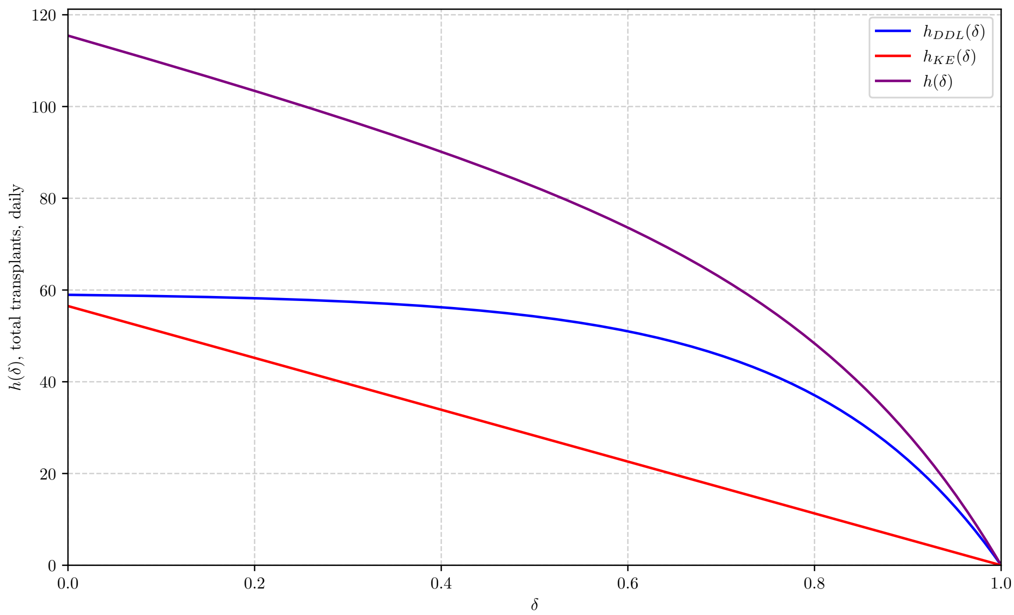

Figure 6: Total transplants, $h(\delta)$, for uniform distribution.

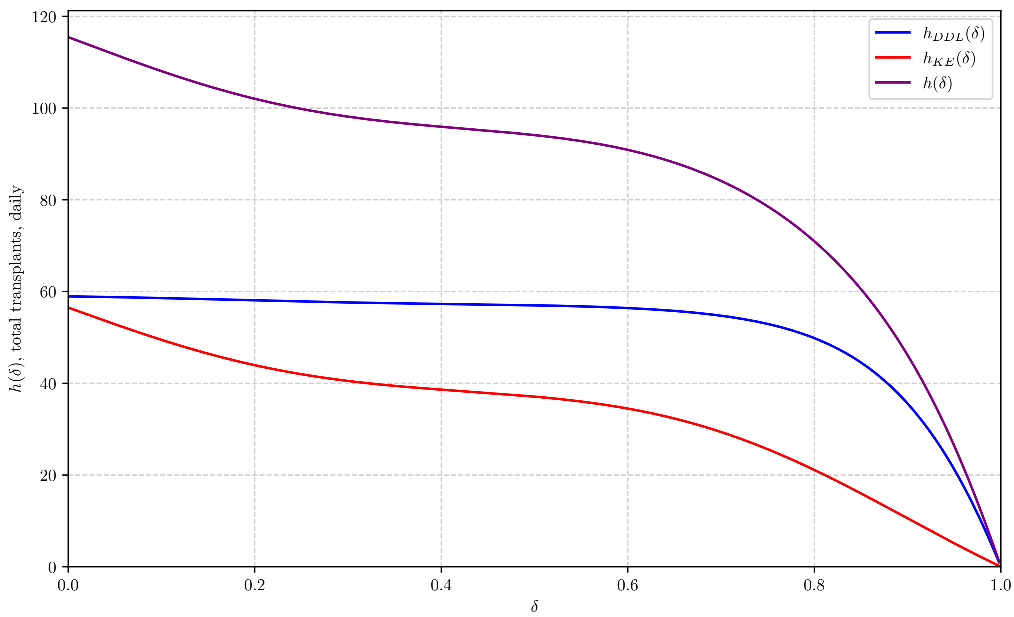

Figure 7: Total transplants, $h(\delta)$, for bimodal distribution.

C Empirical Parameter Discussion

To be completed later as this does not effect the grade. Many parameters were directly found from transplantation data, while others that are not directly reported were derived *from estimates.*

References

Akbarpour, Mohammad, Julien Combe, et al. (Sept. 2024). "Unpaired Kidney Exchange: Overcoming Double Coincidence of Wants without Money". en. In: *Review of Economic Studies*, rdae081. ISSN: 0034-6527, 1467-937X. DOI: 10.1093/restud/rdae081. URL: https://academic.oup.com/restud/advance-article/doi/10.1093/restud/rdae081/7762213 (visited on 12/21/2024).

Akbarpour, Mohammad, Shengwu Li, and Shayan Oveis Gharan (Mar. 2020). "Thickness and Information in Dynamic Matching Markets". In: *Journal of Political Economy* 128.3. Publisher: The University of Chicago Press, pp. 783–815. ISSN: 0022-3808. DOI: 10.1086/704761. URL: https://www.journals.uchicago.edu/doi/abs/10.1086/704761 (visited on 12/23/2024).

Ashlagi, Itai, Maximilien Burq, et al. (June 2019). "On Matching and Thickness in Heterogeneous Dynamic Markets". en. In: *Operations Research*, opre.2018.1826. ISSN: 0030-364X, 1526-5463. DOI: 10.1287/opre.2018.1826. URL: http://pubsonline.informs.org/doi/10.1287/opre.2018.1826 (visited on 12/23/2024).

Ashlagi, Itai, Afshin Nikzad, and Philipp Strack (May 2023). "Matching in Dynamic Imbalanced Markets". en. In: *The Review of Economic Studies* 90.3, pp. 1084–1124. ISSN: 0034-6527, 1467-937X. DOI: 10.1093/restud/rdac044. URL: https://academic.oup.com/restud/article/90/3/1084/6653316 (visited on 12/21/2024).

Ashlagi, Itai and Alvin Roth (Feb. 2021). *Kidney Exchange: An Operations Perspective*. en. Tech. rep. w28500. Cambridge, MA: National Bureau of Economic Research, w28500. DOI: 10.3386/w28500. URL: http://www.nber.org/papers/w28500.pdf (visited on 12/21/2024).

Batabyal, Pikli et al. (Oct. 2012). "Clinical Practice Guidelines on Wait-Listing for Kidney Transplantation: Consistent and Equitable?" en-US. In: *Transplantation* 94.7, p. 703. ISSN: 0041-1337. DOI: 10.1097/TP.0b013e3182637078. URL: https://journals.lww.com/transplantjournal/fulltext/2012/10150/clinical_practice_guidelines_
on_wait_listing_for.7.aspx?casa_token=FAYkIVNNGmoAAAAA:cFJoYfJpyC557G8h37DwCIiyNSM
casa_token=ev0o2vtt2cwAAAAA:7grFRp4ANphvBukkvoK2DTHnev4SDK3GmNOGgYoNnECg7axalPkFis
(visited on 01/05/2025).

Dickerson, John P., Ariel D. Procaccia, and Tuomas Sandholm (Apr. 2019). "Failure-Aware Kidney Exchange". en. In: *Management Science* 65.4, pp. 1768–1791. ISSN:

0025-1909, 1526-5501. DOI: 10.1287/mnsc.2018.3026. URL: https://pubsonline.informs.org/doi/10.1287/mnsc.2018.3026 (visited on 12/23/2024).

Gander, Jennifer C. et al. (Apr. 2020). "Notice of Retraction and Replacement. Gander et al. Association Between Dialysis Facility Ownership and Access to Kidney Transplantation. JAMA. 2019;322(10):957-973". In: *JAMA* 323.15, pp. 1509–1510. ISSN: 0098-7484. DOI: 10.1001/jama.2020.2328. URL: https://doi.org/10.1001/jama.2020.2328 (visited on 01/05/2025).

Ghelichi-Ghojogh, Mousa et al. (Aug. 2022). "The global survival rate of graft and patient in kidney transplantation of children: a systematic review and meta-analysis". In: *BMC Pediatrics* 22, p. 503. ISSN: 1471-2431. DOI: 10.1186/s12887-022-03545-2. URL: https://www.ncbi.nlm.nih.gov/pmc/articles/PMC9404642/ (visited on 01/12/2025).

Health Resources and Services Administration (2024). *Organ Donation Statistics.* https://www.organdonor.gov/learn/organ-donation-statistics. Accessed: 2025-01-16.

OPTN (2024). *Organ Procurement and Transplantation Network.* http://optn.transplant.hrsa.gov. Accessed: 2025-01-16.

— (2025). *OPTN Policies.* Tech. rep. Accessed: 2025-01-16. Health Resources and Services Administration.

Reese, P. P. et al. (2021). "Assessment of the Utility of Kidney Histology as a Basis for Discarding Organs in the United States: A Comparison of International Transplant Practices and Outcomes". In: *Journal of the American Society of Nephrology* 32.2, p. 397. DOI: 10.1681/ASN.2020040464. URL: https://doi.org/10.1681/ASN.2020040464.

Roth, A. E., T. Sonmez, and M. U. Unver (May 2004). "Kidney Exchange". en. In: *The Quarterly Journal of Economics* 119.2, pp. 457–488. ISSN: 0033-5533, 1531-4650. DOI: 10.1162/0033553041382157. URL: https://academic.oup.com/qje/article-lookup/doi/10.1162/0033553041382157 (visited on 12/21/2024).

Stewart, Darren, Tatenda Mupfudze, and David Klassen (Feb. 2023). "Does anybody really know what (the kidney median waiting) time is?" eng. In: *American Journal of Transplantation: Official Journal of the American Society of Transplantation and* *the American Society of Transplant Surgeons* 23.2, pp. 223–231. ISSN: 1600-6143. DOI: 10.1016/j.ajt.2022.12.005.

Ünver, M. Utku (Jan. 2010). "Dynamic Kidney Exchange". In: *The Review of Economic Studies* 77.1, pp. 372–414. ISSN: 0034-6527. DOI: 10.1111/j.1467-937X.2009.00575.x. URL: https://doi.org/10.1111/j.1467-937X.2009.00575.x (visited on 12/21/2024).

USRDS (2023). *2023 USRDS Annual Data Report: Epidemiology of Kidney Disease in the United States*. Tech. rep. Bethesda, MD: National Institutes of Health, National Institute of Diabetes, Digestive, and Kidney Diseases.